SPHERICAL FLOW OF A VISCOUS HEAT-CONDUCTING GAS INTO AN OCCUPIED SPACE

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The spherical flow of a gas into a space where the pressure is not zero is studied on the basis of the Navier–Stokes equations. The constraint problem is solved by the relaxation method. This method enables one to analyze the effects of the Reynolds number, of the pressure ratio, of the sonic sphere to surrounding space, and also of the temperature in the surrounding space on the distribution of parameters within the flow region. It is shown, in particular, that, for a constant Reynolds number and variable pressure ratio, the expansion front of a viscous heat-conducting gas in vacuum will be the envelope of the parameter distributions within the supersonic region. The results of these calculations are compared with those of other authors.

The effect of viscosity and thermal conductivity on the spherical flow of a gas into occupied space was analyzed in several theoretical studies. In most of them the method of a small parameter [1, 2] was applied to solve the Navier–Stokes equations. With this method it was possible to perform a qualitative and, in some cases, also a quantitative analysis for large Reynolds numbers R_*.

The problems of a viscous heat-conducting gas flowing into vacuum were studied by Ladyzhenskii [3].

Apparently, the first complete enough solution to the problem was arrived at by Gusev and Zhbakova [4]. The idea behind the method used by these authors was to convert the constraint problem into a Cauchy problem. The initial conditions were found by a series expansion around a point at infinity $r = \infty$ and a subsequent integration in the upstream direction.

Obviously, this method imposes no restrictions on the values of the R_* number or the pressure ratio p_*/p_∞. However, every series expansion which is specific in terms of the coefficients establishes a definite relation between R_* and p_*/p_∞, so that calculation, e.g., for a constant R_* and variable p_*/p_∞, will require different series expansions with the choice of coefficients for every p_*/p_∞ not obvious. It now becomes difficult to analyze the effect of the Reynolds number and that of the pressure ratio on the flow structure.

In this article the authors propose another approach: when a conversion is made into thermodynamic variables, that peculiar situation at $r \to \infty$ does not arise, and the constraint problem can be solved by conventional methods, e.g., by the relaxation method. The analysis of the flow pattern is hereby considerably simplified.

As in most preceding studies, the analysis is performed for the case where the limiting pressure in the gas is much higher than the pressure in the surrounding space. This case is the most interesting one for both theoretical and practical considerations [2, 4].

The spherical expansion of a gas at a high pressure ratio can be realized in practice [5], but not sufficient experimental data are available yet for a comparison with theoretical results.

Statement of the Problem. The spherical flow of a viscous heat-conducting gas is described by the system of equations:

$$\rho u \frac{du}{dr} + \frac{1}{M_*^2}\frac{dp}{dr} = \frac{1}{R_*}\left\{\frac{4}{3}\frac{d}{dr}\left[\frac{\mu}{r^2}\frac{d}{dr}(r^2 u)\right] - \frac{4u}{r}\frac{d\mu}{dr}\right\} \quad (1)$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 122-126, May-June, 1971. Original article submitted March 2, 1970.

$$\rho u \frac{dT}{dr} - (\gamma - 1) u \frac{dp}{dr} = \frac{1}{R_*}\left\{\frac{1}{\sigma}\frac{1}{r^2}\frac{d}{dr}\left(\mu r^2 \frac{dT}{dr}\right) - \frac{4}{3}(\gamma - 1)\mu M_*^2 \left[\frac{d}{dr}\left(\frac{u^2}{r}\right) - \left(\frac{du}{dr}\right)^2\right]\right\} \tag{2}$$

$$\rho u r^2 = 1, \qquad p = \frac{1}{\gamma}\rho T \tag{3}$$

Here σ is the Prandtl number and $\mu = \mu(T)$ is the viscosity as some function of temperature. The flow parameters in (1), (2), and (3) are dimensionless quantities:

$$\begin{gathered} r = \frac{r'}{r_*'}, \quad u = \frac{u'}{u_*'}, \quad T = \frac{T'}{T_*'}, \quad \rho = \frac{\rho'}{\rho_*'}, \quad p = \frac{p'}{\rho_*' u_*'^2} \\ \mu = \frac{\mu'}{\mu_*'}, \quad R_* = \frac{\rho_*' u_*' r_*'}{\mu_*'}, \quad M_* = \frac{u_*'}{\sqrt{\gamma' R' T_*'}}, \quad \sigma = \frac{\mu_*' c_p'}{k_*'} \end{gathered} \tag{4}$$

The Prandtl number σ and the ratio of specific heats γ are assumed constant.

Let the gas flow into a space where the pressure is not zero and the temperature is finite. There always exists a surface here with some $r = r_1$ behind which the gas flow is entirely subsonic: $M_1^2 \ll 1$. It follows from a consideration of the heat transfer equation (2) for $r > r_1$ that the condition of a finite temperature at $r \to \infty$, i.e., $dT/dr = 0$, alone is not sufficient for a spherical flow but that the temperature at infinity must be specified. Two thermodynamic parameters may thus have arbitrarily assigned values at infinity, i.e., in the surrounding space.

The state parameters of the gas may also have specified values at the source surface. With R_* and M_* given, the flow rate and r_*' will then be defined.

It is not convenient to use the constraint with respect to velocity, since $u(\infty)$ is a singular point [2].

The boundary conditions will be stated as

$$\rho = 1,\ T = 1 \ \text{for} \quad r = 1,\ \rho \to \rho_\infty,\ T \to T_\infty \ \text{for}\ r \to \infty \tag{5}$$

Method of Solution. After eliminating the velocity and the pressure from Eqs. (1), (2), and (3), and performing the $r = 1 + \alpha \tan(\frac{1}{2}\pi y)$ transformation, we will rewrite the system of equations in a form convenient for subsequent linearization

$$\begin{gathered} A_1 \frac{d^2\rho}{dy^2} + B_1 \frac{d\rho}{dy} + C_1\rho = D_1 \\ A_2 \frac{d^2T}{dy^2} + B_2 \frac{dT}{dy} + C_2 T = D_2 \end{gathered} \tag{6}$$

$$\rho = 1,\ T = 1 \ \text{for}\ y = 0, \quad \rho = \rho_\infty,\ T = T_\infty \ \text{for} \quad y = 1 \tag{7}$$

Here α is some constant and A_k, B_k, C_k, and D_k are factors which remain after extraction of the linear part and which generally depend on y, ρ, T, $d\rho/dy$, dT/dy, and R_*, σ, M_*, γ, α.

The problem (6), (7) has been solved by the relaxation method according to an implicit procedure. The space grid was uniform with respect to y, with N subdivisions and the interval h.

The linearized system

$$\begin{gathered} \rho_i^j \left(\frac{\partial\rho}{\partial t}\right)_i^{j+1} + A_{1i}^j \left(\frac{\partial^2\rho}{\partial y^2}\right)_i^{j+1} + B_{1i}^j \left(\frac{\partial\rho}{\partial y}\right)_i^{j+1} + C_{1i}^j \rho_i^{j+1} = D_{1i}^j \\ \rho_i^j \left(\frac{\partial T}{\partial t}\right)_i^{j+1} + A_{2i}^j \left(\frac{\partial^2 T}{\partial y^2}\right)_i^{j+1} + B_{2i}^j \left(\frac{\partial T}{\partial y}\right)_i^{j+1} + C_{2i}^j T_i^{j+1} = D_{2i}^j \\ \rho_0^j = 1, \quad \rho_N^j = \rho_\infty, \qquad T_0^j = 1, \quad T_N^j = T_\infty \end{gathered}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_i^{j+1} = \frac{f_i^{j+1} - f_i^j}{\tau}, \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_i^{j+1} = \frac{f_{i+1}^{j+1} - 2f_i^{j+1} + f_{i-1}^{j+1}}{h^2}, \quad \left(\frac{\partial f}{\partial y}\right)_i^{j+1} = \frac{f_{i+1}^{j+1} - f_{i-1}^{j+1}}{2h}$$

was solved for each j + 1 iteration by the method of successive approximations [6, 7]. The magnitude of the iteration parameter τ remained unchanged during the relaxation process.

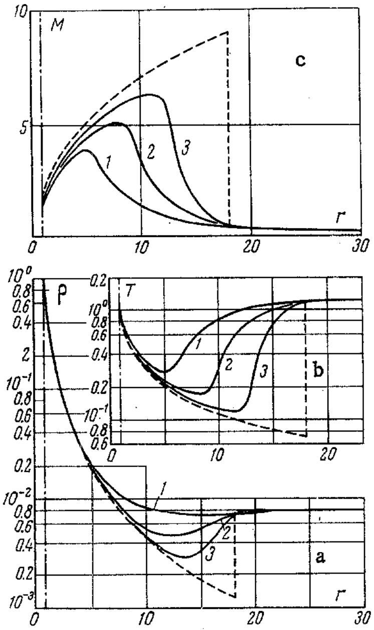

Fig. 1

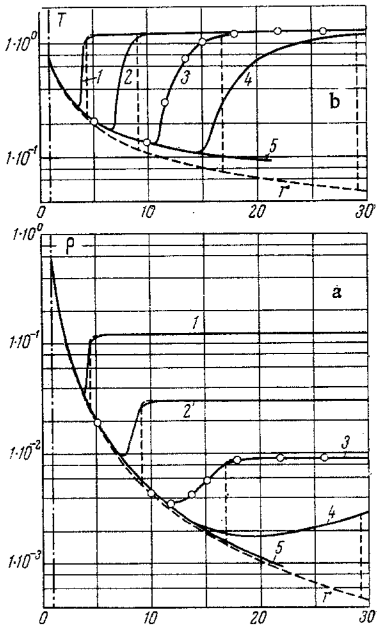

Fig. 2

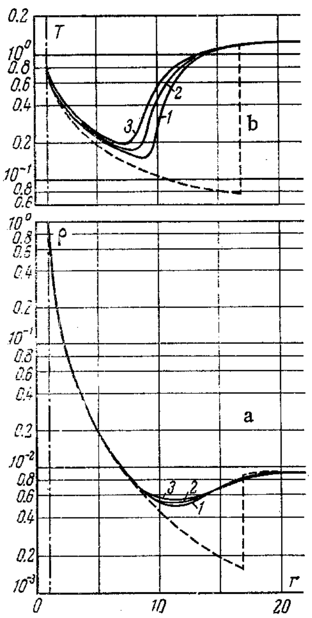

Fig. 3

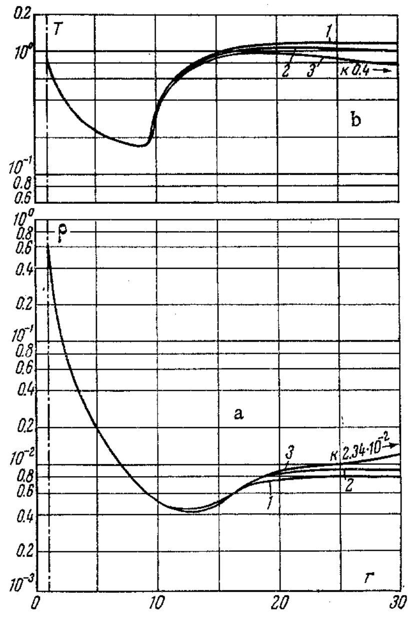

Fig. 4

The initial approximation $\rho_i^\circ = \varphi_1(i)$, $T_i^\circ = \varphi_2(i)$, as a rule, was chosen in the following manner: the density for $0 \leq i \leq i_0$ was calculated from the equations of isentropy and for $i_0 < i \leq N$ was assumed equal to ρ_∞ (i_0 is the last point at which the density corresponding to isentropic expansion is not lower than ρ_∞),

$$T_i^\circ = T_\infty - (\gamma - 1)/2\,(\rho_i^\circ r_i^2)^2$$

for

$$0 \leqslant i \leqslant N$$

The computation was terminated after

$$\max_i\left[\left|\left(\frac{\partial \ln \rho}{\partial t}\right)_i^{j+1}\right|, \quad \left|\left(\frac{\partial \ln T}{\partial t}\right)_i^{j+1}\right|\right] \leqslant \varepsilon_0$$

had been reached.

The values of τ, h, and ε_0 were chosen by control calculations. With $R_* = 100$ and $p_*/p_\infty = 92.7$, for example, $h = 2 \cdot 10^{-3}$, $\varepsilon_0 = 2 \cdot 10^{-3}$, $\tau = 5 \cdot 10^{-2}$, and the solution stabilized after 500 iterations.

Discussion of Results. The following structure evolves during the spherical flow of an ideal gas into a space where the pressure is not zero, $p_\infty < p_{0*}$: while expanding isentropically from some sonic or supersonic surface, the gas overexpands until the pressure becomes lower than p_∞. After that, there takes place a limiting of the gas in a shock wave manner accompanied by a recovery pressure and then a nonisentropical limiting down to the parameters of the occupied space. The position of the shock wave is determined from the ratio p_*/p_∞. We note that the ideal solution can be obtained only when $T_\infty = T_{0*}$.

In Figs. 1, 2, and 3 are shown typical calculated results illustrating the flow of a viscous heat-conducting gas from a sonic surface $M_* = 1$. The temperature dependence of viscosity was accounted for by using the Sutherland equation:

$$\mu = T^{3/2}\frac{1+T_s}{T+T_s}$$

Calculations were made for air ($\gamma = 1.4$, $T_s' = 104°K$) with a temperature limiting at the sonic surface $T_{0*}' = 293°K$.

The distribution of flow parameters p, T, and M for $p_*/p_\infty = 109$, $T_\infty/T_* = 1.2$, and $\sigma = 0.7$ is shown in Fig. 1. (Curves 1, 2, and 3 correspond to $R_* = 50$, 100, and 200; the dashed curve represents the solution for an ideal gas.) Besides the evident regularities, one also ought to note that the density curves intersect within a small region around the point corresponding to the density value behind the shock wave in the idealized case.

The departure of the parameters from their isentropic distributions begins practically at the sonic surface.

Calculations made for constant R_* numbers and variable p_*/p_∞ ratios (specifically for $R_* = 161.83$, $T_\infty/T_* = 1.2$, $\sigma = 0.7$ shown in Fig. 2, where curves 1, 2, 3, and 4 correspond to $p_*/p_\infty = 6.94$, 27.8, 92.7, and 278) indicate that the departure from isentropic conditions in most of the supersonic region is not related to the position of the shock wave, but is determined only by the R_* number, i.e., that the natural effect of the shock wave on the upstream flow is localized. (The dashed curves in Fig. 2 represent the solution for an ideal gas, the small circles mark the points determined in [4].)

The distance from the ideal position of the shock wave to the point of lowest temperature will be referred to the free path length within it. In the modes analyzed so far the ratio of the two lengths has been on the order of 6-9 and has tended to increase with higher Mach numbers "ahead" of the shock wave. This ratio may, in a certain way, serve as a characterization of the shock wave width in terms of free "confluent flow" lengths.

The effect of the shock wave on the upstream flow is evidently also determined by the value of the σ number. This can be seen in Fig. 3, where curves 1, 2, and 3 correspond to flow numbers $\sigma = 1.0$, 0.7, and 0.5, respectively. Calculations were made here for $R_* = 100$, $T_\infty/T_* = 1.2$, $p_*/p_\infty = 92.7$.

With a constant σ, then, the distribution of parameters within the region adjacent to the sonic sphere and bounded for every value of p_*/p_∞ by the effect of the shock wave setting in is, in this way, determined only by the R_* number and does not depend on the pressure in the surrounding space. The results shown in Fig. 4 (where curves 1, 2, and 3 correspond to $T_\infty/T_* = 1.2$, 1.0, and 0.4, respectively, and where calculations have been made for $R_* = 100$, $p_*/p_\infty = 109$, $\sigma = 0.7$) indicate that also the effect of temperature in the surrounding space does not extend to this region. One may assume that this situation is maintained at $p_\infty \to 0$ ($p_*/p_\infty \to \infty$ for $R_* = \text{const}$). The envelopes of the parameter distribution curves (curves 5 in Fig. 2) will then correspond to the initial phase of expansion of a viscous heat-conducting gas into vacuum.

The comment concerning the temperature is of essential significance, generally speaking, because during the flow into vacuum with $r \to \infty$ the assumption of no momentum transfer and no heat flow into the surrounding medium can be made only if there is a thermal interaction between gas molecules, and then T_∞ may not be equal to T_{0*}. Indeed, it follows from the analysis in [3] that during a spherical flow into vacuum with $r \to \infty$ one has $u \to 0$ and, consequently, $T_{0\infty} \to T_\infty$. For finite R_* numbers the temperature limiting in the flow region cannot be sustained, and $T_{0\infty}$ may differ from T_{0*}; this last effect is clearly observed in the example of a cylindrical source [8].

Calculations for $R_* = 161.83$, for example, were performed with p_*/p_∞ up to about 10^8, which almost corresponds to a flow into deep vacuum.

The positions of the envelopes (they will be called curves of viscous expansion) are determined by the conditions on the sonic surface as well as by the values of the R_* and the σ numbers. The dependence on the σ number is, as Fig. 3 shows, rather weak.

Analogous curves were obtained earlier in [2]. The fact that the parameter distributions within the region up to the shock wave do not depend on the conditions in the surrounding space was not deduced from the analysis, however, but was stipulated in order to make it possible to plot the entire flow region.

In extending the Navier–Stokes equations to a gas flowing into vacuum, or practically into a space where the pressure is very low, the question of their validity here will naturally arise. This question has been treated thoroughly in [3].

The authors express their gratitude to M. A. Gol'dshtik and B. G. Kuznetsov for valuable comments.

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